Lecture 15: Universal Hashing: Minimizing Collisions

Recall I

- k-wise Independence
  - Intuition: First k inputs are answered uniformly at random
  - Formally: For all distinct $x_1, \dots, x_k \in \mathcal{D}$ and $y_1, \dots, y_k \in \mathcal{R}$ we have

$$\mathbb{P}\left[h(x_1) = y_1, h(x_2) = y_2, \dots, h(x_k) = y_k \colon h \xleftarrow{\$} \mathcal{H}\right] = \frac{1}{|\mathcal{R}|^k}$$

  - One Construction: The set of all degree $< k$ polynomials.

Recall II

- 2-wise Independence/Pairwise Independence
  - Special case of $k = 2$ mentioned above
  - Formally: For all distinct $x_1, x_2 \in \mathcal{D}$ and $y_1, y_2 \in \mathcal{R}$ we have

$$\mathbb{P}\left[h(x_1) = y_1, h(x_2) = y_2 \colon h \stackrel{\$}{\leftarrow} \mathcal{H}\right] = \frac{1}{\left|\mathcal{R}\right|^2}$$

  - One Construction: Linear functions

Recall III

- Universal Hash Function Family
  - Intuition: Probability of Collision is low
  - Formally: For all distinct $x_1, x_2 \in \mathcal{D}$ we have

$$\mathbb{P}\left[h(x_1) = h(x_2) \colon h \xleftarrow{\$} \mathcal{H}\right] \leqslant \frac{1}{|\mathcal{R}|}$$

  - Construction: Any 2-wise independent hash function family is also universal (we proved this result). The collision probability $\mathbb{P}\left[h(x_1) = h(x_2) \colon h \xleftarrow{\$} \mathcal{H}\right] = \frac{1}{|\mathcal{R}|}$ in this case.

Recall IV

- Constructing Better Universal Hash Function Families
  - We know that if the range is larger (or same size) than the domain, then we can achieve collision probability $\mathbb{P}\left[h(x_1) = h(x_2) \colon h \xleftarrow{\$} \mathcal{H}\right] = 0$ for every distinct $x_1, x_2 \in \mathcal{D}$ (we saw that any one-one function achieves this)
  - When the range is smaller than the domain, we saw that any 2-wise independent hash function family achieves collision probability $\mathbb{P}\left[h(x_1) = h(x_2) \colon h \xleftarrow{\$} \mathcal{H}\right] = \frac{1}{|\mathcal{R}|}$
  - When the range is smaller than the domain, can we have collision probability $\mathbb{P}\left[h(x_1) = h(x_2) \colon h \xleftarrow{\$} \mathcal{H}\right] < \frac{1}{|\mathcal{R}|}$ for all distinct $x_1, x_2 \in \mathcal{D}$?
    - In the previous lecture we saw that we can construct one hash function family $\mathcal{H}$, for $|\mathcal{D}| = 4$, $|\mathcal{R}| = 2$ such that the collision probability is $= \frac{1}{3} < \frac{1}{|\mathcal{R}|} = \frac{1}{2}$!
    - Can we have even lower collision probabilities? In this lecture we shall prove that a lower collision probability is impossible!

Lower-bounding Collision Probability

- Let the size of the domain $\mathcal{D}$ be N
- Let the size of the range $\mathcal{R}$ be M
- Suppose we have $M < N$

We shall prove the following theorem

Theorem (Collision Lower Bound)

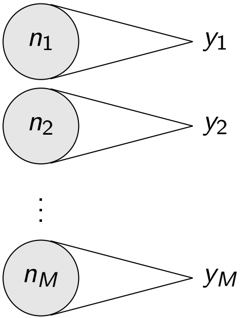
Let $\mathcal{H}$ be a hash function family such that the domain of the function is $\mathcal{D}$ and the range of the functions is $\mathcal{R}$. There exists distinct $x_1^*, x_2^* \in \mathcal{D}$ such that

$$\mathbb{P}\left[h(x_1^*) = h(x_2^*) \colon h \xleftarrow{\$} \mathcal{H}\right] \geqslant \frac{\frac{N}{M} - 1}{N - 1}$$

Note that for $M = 2$ and $N = 4$, the bound is $1/3$. The has function family from the previous lecture achieves this bound.

Proof of the Lower-bound I

- Let us fix a hash function $h \in \mathcal{H}$
- Suppose the range is the set $\{y_1, y_2, \dots, y_M\}$
- Let n_i be the size of the set $\{x : x \in \mathcal{D}, h(x) = y_i\}$, for $i \in \{1, 2, \dots, M\}$. That is, n_1 inputs maps to y_1, n_2 inputs maps to y_2, and so on ...
- The intuition of this is pictorially represented below



Proof of the Lower-bound II

- Let us count the number (represented by $\#\mathrm{col}_h$) of entries $\{x_1, x_2\}$, where x_1, x_2 are distinct elements from the domain $\mathcal{D}$, such that $h(x_1) = h(x_2)$

Claim

$$\#\mathrm{col}_h = \sum_{i=1}^{M} \binom{n_i}{2}$$

Proof.

- Note that the number of distinct $\{x_1, x_2\}$ that collide at y_1 is $\binom{n_1}{2}$
- Note that the number of distinct $\{x_1, x_2\}$ that collide at y_2 is $\binom{n_2}{2}$
- And, so on ...
- Adding these entries, we get the total number of distinct $\{x_1, x_2\}$ that collide $\square$

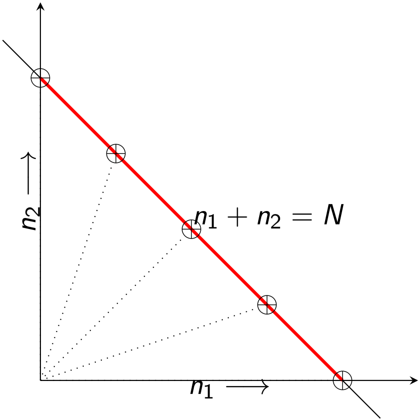
Proof of the Lower-bound III

- Note that $n_i \geqslant 0$ and $\sum_{i=1}^{M} n_i = N$
- We are interested in lower-bounding the expression $\sum_{i=1}^{M} \binom{n_i}{2}$
- Consider the following manipulation

$$\begin{aligned}
\sum_{i=1}^{M} \binom{n_i}{2} &= \sum_{i=1}^{M} \frac{n_i(n_i - 1)}{2} \\
&= \sum_{i=1}^{M} \frac{n_i^2 - n_i}{2} \\
&= \sum_{i=1}^{M} \frac{n_i^2}{2} - \frac{N}{2}
\end{aligned}$$

Proof of the Lower-bound IV

- We are interested in lower-bounding $\sum_{i=1}^{M} n_i^2$ under the constraint $n_i \geqslant 0$ and $\sum_{i=1}^{M} n_i = N$
- So our task is to look at all the solutions to the equations: $n_i \geqslant 0$ (for all $i \in \{1, \ldots, M\}$) and $\sum_{i=1}^{M} n_i = N$. And minimize $\sum_{i=1}^{M} n_i^2$.
- For $M = 2$, we have the following picture for intuition. The THICK RED line is the set of all feasible solutions. The quantity $n_1^2 + n_2^2$ measures the distance of the solution from the origin. The dotted lines represent this distance for various solutions.
- Using the AM-GM inequality, one can show that the minimum is achieved when all the coordinates of the solution are equal.



$n_1 + n_2 = N$
$n_2 \longrightarrow$
$n_1 \longrightarrow$

- So, the solution where $n_1 = n_2 = \cdots = n_M$ and $\sum_{i=1}^{M} n_i = N$ is

$$\left(\frac{N}{M}, \frac{N}{M}, \ldots, \frac{N}{M}\right)$$

- For this feasible solution, we have:

$$\sum_{i=1}^{M} n_i^2 = \sum_{i=1}^{M} (N/M)^2 = N^2/M$$

- Therefore, we get

Claim

$$\#\mathrm{col}_h \geqslant \frac{\frac{N^2}{M} - N}{2}$$

Proof of the Lower-bound VII

- Suppose $\mathcal{H} = \{h_1, \dots, h_K\}$. Then, the total number (represented by $\#\mathrm{col}_{\mathcal{H}}$) of entries $\{h, x_1, x_2\}$, where x_1, x_2 are distinct elements from the domain $\mathcal{D}$, $h \in \mathcal{H}$, and $h(x_1) = h(x_2)$ is

Claim

$$\#\mathrm{col}_{\mathcal{H}} \geqslant K \left(\frac{\frac{N^2}{M} - N}{2} \right)$$

Proof.

- For each h, we have shown earlier that $\#\mathrm{col}_h \geqslant \left(\frac{\frac{N^2}{M} - N}{2} \right)$.
- Summing over all $h \in \mathcal{H}$, we get this result $\square$

Proof of the Lower-bound VIII

- Let us define $\mathcal{P}$ be the set of all distinct $\{x_1, x_2\}$ such that $x_1, x_2 \in \mathcal{D}$. Note that $|\mathcal{P}| = \binom{N}{2} = N(N-1)/2$
- Suppose we perform the following experiment:

  1. Sample $(x_1, x_2) \xleftarrow{\$} \mathcal{P}$
  2. Sample $h \xleftarrow{\$} \mathcal{H}$
  3. Output 1 if $h(x_1) = h(x_2)$; otherwise output 0

  Let us denote the output of this experiment by Z.
- Let us calculate expected outcome of Z

- Consider the following manipulation

$$
\begin{aligned}
\mathbb{E}\left[Z\colon (x_1,x_2)\xleftarrow{\$}\mathcal{P}, h\xleftarrow{\$}\mathcal{H}\right] &= \mathbb{P}\left[Z=1\colon (x_1,x_2)\xleftarrow{\$}\mathcal{P}, h\xleftarrow{\$}\mathcal{H}\right]\\
&= \frac{\#\mathrm{col}_{\mathcal{H}}}{|\mathcal{P}|\cdot|\mathcal{H}|}\\
&\geqslant \frac{K\left(\frac{\frac{N^2}{M}-N}{2}\right)}{\frac{N(N-1)}{2}\cdot K}\\
&= \frac{\frac{N}{M}-1}{N-1}
\end{aligned}
$$

Proof of the Lower-bound X

- So, we get the following result

Claim

$$\mathbb{E}\left[Z\colon (x_1, x_2) \xleftarrow{\$} \mathcal{P}, h \xleftarrow{\$} \mathcal{H}\right] \geqslant \frac{\frac{N}{M} - 1}{N - 1}$$

- Note that the above expression is identical to the following statement:
  For $(x_1, x_2) \xleftarrow{\$} \mathcal{P}$, we have $\mathbb{E}\left[Z\colon h \xleftarrow{\$} \mathcal{H}\right] \geqslant \frac{\frac{N}{M}-1}{N-1}$
- By Pigeon-hole Principle, we get: There exists $(x_1^*, x_2^*) \in \mathcal{P}$ such that

$$\mathbb{E}\left[Z\colon h \xleftarrow{\$} \mathcal{H}\right] \geqslant \frac{\frac{N}{M} - 1}{N - 1}$$

- So, for this choice of x_1^* and x_2^* the collision probability is

$$\mathbb{P}\left[h(x_1^*) = h(x_2^*) \colon h \xleftarrow{\$} \mathcal{H}\right] \geqslant \frac{\frac{N}{M} - 1}{N - 1}$$

- This completes the proof of the theorem

"Best Universal Hash Functions"

- Given domain of size N and range of size M, where $M < N$ and M divides N
- Can we design universal hash functions such that for all distinct $x_1, x_2 \in \mathcal{D}$ we have

$$\mathbb{P}\left[h(x_1) = h(x_2) \colon h \xleftarrow{\$} \mathcal{H}\right] = \frac{\frac{N}{M} - 1}{N - 1} = \frac{1}{M} \cdot \frac{N - M}{N - 1}$$

- This implies that we have to achieve equality at every step of the proof of the collision lower-bound theorem
    - We have to ensure $n_1 = n_2 = \cdots = n_M$
    - We have to ensure that the "average" collision probability for every (x_1, x_2) is identical
- This problem will be posed in the homework

"Better(?) than k-wise Independence"

- Note that when defining k-wise Independence we stated that the probability of a hash function mapping $x_1 \mapsto y_1$, $x_2 \mapsto y_2$, ..., and $x_k \mapsto y_k$ is
$$= \frac{1}{|\mathcal{R}|^k}$$
- Why did we not write $\leqslant \frac{1}{|\mathcal{R}|^k}$?
- Is it even possible to get $< \frac{1}{|\mathcal{R}|^k}$?
- In the homework you will prove that for any hash function family, there exists distinct $x_1, \dots, x_k$ and $y_1, \dots, y_k$ such that
$$\mathbb{P}\left[h(x_1) = y_1, \dots, h(x_k) = y_k \colon h \xleftarrow{\$} \mathcal{H}\right] \geqslant \frac{1}{|\mathcal{R}|^k}$$
- So, there is no way to get $< \frac{1}{|\mathcal{R}|^k}$. The bound $\leqslant \frac{1}{|\mathcal{R}|^k}$ would be equivalent to the bound $= \frac{1}{|\mathcal{R}|^k}$.

Appendix: Inequality Proof I

Suppose $n_1, \ldots, n_M$ are positive numbers such that $n_1 + \cdots + n_M = N$. Then the following claim holds.

Claim

$$n_1^2 + \cdots + n_M^2 \geqslant N^2/M$$

Proof.

- We shall use AM-GM inequality to prove this result
- AM-GM inequality states that, for non-negative a and b, the following holds.

  $$\frac{a+b}{2} \geqslant \sqrt{ab}$$

  Moreover, the equality holds if and only if $a = b$.

Appendix: Inequality Proof II

- Consider the following manipulation of the original expression

$$
\begin{aligned}
\sum_{i=1}^{M} n_i^2 &= (n_1 + \cdots + n_M)^2 - \sum_{1 \leqslant i < j \leqslant M} 2 n_i n_j \\
&= N^2 - \sum_{1 \leqslant i < j \leqslant M} 2 n_i n_j, && \text{Using } \sum_{i=1}^{M} n_i = N \\
&\geqslant N^2 - \sum_{1 \leqslant i < j \leqslant M} (n_i^2 + n_j^2), && \text{Using AM-GM} \\
&= N^2 - (M-1) \sum_{1 \leqslant i \leqslant M} n_i^2
\end{aligned}
$$

- Rearranging, we get

$$M \sum_{i=1}^{M} n_i^2 \geqslant N^2$$

- This gives the inequality of the claim. Equality holds if and only if $n_i = n_j$, for all $1 \leqslant i < j \leqslant M$. This holds if and only if $n_1 = n_2 = \cdots = n_M$